Solitons for the rotating reduced Maxwell-Bloch equations with anisotropy

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For the reduced Maxwell-Bloch equations with two components of polarization and with an anisotropic dipole moment we establish a hierarchy of soliton solutions by use of Bäcklund transformations. The *N*-soliton formulas are given in terms of Vandermonde-like determinants. Differences in the respective solution manifolds for the so-called self-induced transparency (SIT) equations and three types of reduced Maxwell-Bloch equations are pointed out.

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I. INTRODUCTION

A little more than thirty years ago [1–5] the reduced Maxwell-Bloch (RMB) equations were established and used to get for the phenomenon of self-induced transparency (SIT) a more accurate description compared to the so-called SIT equations, i.e., to go beyond the slowly varying envelope (SVEA) approximation. The original treatment of the RMB equations as an integrable system has been restricted to plane polarized waves, and it has been claimed that the RMB equations for circularly polarized light or rotating RMB equations "do not have soliton solutions and are not an integrable system" ([4], p. 374). However, it was found in [6] that these equations are indeed integrable by using the inverse scattering transform method and that they are connected with a scattering problem of Kaup-Newell type [7] being quadratic in the spectral parameter.

In the present paper we are dealing with more general rotating RMB equations where the effective dipole moment of the considered atomic transition is anisotropic. On the other hand, the velocity c of light as it appears in the Maxwell equations (1), see below, is assumed to be the same for both field components, i.e., we are considering anisotropic molecules in an isotropic medium. The RMB equations generalized in this way are a new integrable system with remarkable structural properties. It seems that the related spectral problem has not yet been investigated in the literature.

The RMB equations with anisotropy are derived and prepared in Sec. II. In Sec. III the crucial system of simultaneous linear differential equations is established, and some symmetries and conservation laws are discussed in this section. A Darboux transformation is derived in Sec. IV and extended to become a Bäcklund transformation in Sec. V. For the sake of clarity we repeat that a *Darboux transformation* is a transformation of some *scattering* or *spectral problem* [here defined by the first of equations (15)] while a *Bäcklund transformation* applies to a simultaneous system [here both equations (15)]. The *N*-fold Bäcklund transformation is established in Sec. VI. Here we make use of a compact notation by use of *Vandermonde-like determinants* which are defined in the Appendix more generally than hitherto. As example solutions we discuss in Sec. VII a simple soliton and the breather. Summary and conclusions are given in Sec. VII. There, in particular, we compare three different types of RMB equations with the so-called self-induced transparency (SIT) equations and point out structural differences in the respective soliton hierarchies. It is a common feature of all the versions of RMB equations considered here that particular breather solutions correspond to the 2π pulse of the SIT equations and, therefore, to the solitons found in the SIT experiments [8–10].

II. THE REDUCED MAXWELL-BLOCH EQUATIONS WITH ANISOTROPY

Just as we did in the foregoing paper [6] we consider the interaction between an optical wave propagating in z direction and an atomic two-level system with a dipole transition $\Delta J=0, \Delta M=1$ with J and M denoting the total angular momentum and its z component, respectively. Now we assume an anisotropy insofar as two dipole moments $d_x \neq d_y$ instead of one appear in the Maxwell-Bloch equations, and the reduced Maxwell-Bloch equations take the form

$$(c\partial_z+\partial_t)\mathcal{E}_x=-2\pi d_x n\partial_t\mathcal{R}_x,$$

$$(c\partial_z+\partial_t)\mathcal{E}_y=-2\pi d_y n\partial_t\mathcal{R}_y, \tag{1}$$

$$\partial_t\mathcal{R}_x=-\omega_0\mathcal{R}_y-\frac{2d_y}{\hbar}\mathcal{E}_y\mathcal{R}_z,$$

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$$\partial_t \mathcal{R}_y = \omega_0 \mathcal{R}_x + \frac{2d_x}{\hbar}\mathcal{E}_x \mathcal{R}_z,$$

$$\partial_t \mathcal{R}_z = \frac{2}{\hbar}(d_y \mathcal{E}_y \mathcal{R}_x - d_x \mathcal{E}_x \mathcal{R}_y). \tag{2}$$

Here $\mathcal{E}_x$ and $\mathcal{E}_y$ are the electric field components, $(\mathcal{R}_x, \mathcal{R}_y, \mathcal{R}_z)$ is the Bloch vector, c is the velocity of light in the host medium, ω_0 is the resonance frequency, and n denotes the number density of atoms. The Bloch equations (2) are equivalent to equations (2.22) in [11]. The approximation inherent to all variants of the RMB equations can be understood as follows. Let us start from the full Maxwell equations and here, e.g., write down only the equation for the x component of the field,

$$(\partial_t^2 - c^2\partial_z^2)\mathcal{E}_x \equiv (\partial_t - c\partial_z)(\partial_t + c\partial_z)\mathcal{E}_x = 4\pi d_x n \partial_t^2 \mathcal{R}_x. \tag{3}$$

Under the assumption that the righthand side is much smaller than $|\partial_t^2 \mathcal{E}_x|$ and that initially there is no left-going wave we may replace $(\partial_t - c\partial_z)$ by $2\partial_t$. A first integration then leads to the first equation of (1). It has been estimated [1,2] that this approximation works well as long as $n < 10^{18}\ \mathrm{cm}^{-3}$. For example, in the classical SIT experiments of Gibbs and Slusher [9] n is of the order $10^{12}\ \mathrm{cm}^{-3}$.

Now we take characteristic coordinates χ, τ_0 together with a proper scaling,

$$\chi = (4\pi d_x d_y n/\hbar c)z, \quad \tau_0 = \omega_0(t - z/c), \tag{4}$$

$$\widetilde{E}_{x,y}(\chi, \tau_0) = (2\sqrt{d_x d_y}/\hbar\omega_0)\mathcal{E}_{x,y}(z,t), \tag{5}$$

$$\widetilde{R}_{x,y,z}(\chi, \tau_0) = \mathcal{R}_{x,y,z}(z,t). \tag{6}$$

Let us write $f = d_y/d_x$ and assume $f \geqslant 1$, without loss of generality. Then Eqs. (1) and (2) may be rewritten as

$$\partial_\chi \widetilde{E}_x = \widetilde{R}_y/\sqrt{f} + \widetilde{E}_y \widetilde{R}_z,$$

$$\partial_\chi \widetilde{E}_y = -\widetilde{R}_x\sqrt{f} - \widetilde{E}_x \widetilde{R}_z,$$

$$\partial_{\tau_0} \widetilde{R}_x = -\partial_\chi \widetilde{E}_x \sqrt{f},$$

$$\partial_{\tau_0} \widetilde{R}_y = -\partial_\chi \widetilde{E}_y/\sqrt{f},$$

$$\partial_{\tau_0} \widetilde{R}_z = \widetilde{R}_x \widetilde{E}_y \sqrt{f} - \widetilde{R}_y \widetilde{E}_x/\sqrt{f}. \tag{7}$$

Clearly, for $f=1$ we recover the isotropic system.

We define the numbers

$$f_\pm = (f \pm f^{-1})/2, \quad a = f_-/(2f_+), \tag{8}$$

rescale the retarded time variable,

$$\tau = f_+ \tau_0 \tag{9}$$

and write

$$R_{x,y,z}(\chi, \tau) = \widetilde{R}_{x,y,z}(\chi, \tau_0), \tag{10}$$

$$E_{x,y}(\chi, \tau) = \widetilde{E}_{x,y}(\chi, \tau_0), \tag{11}$$

We introduce complex fields,

$$E = (E_x + iE_y)/\sqrt{f_+},$$

$$R = (R_x\sqrt{f} + iR_y/\sqrt{f})/\sqrt{f_+}, \tag{12}$$

and write down a more general system,

$$\partial_\chi E = -i(R + ER_z),$$

$$\partial_\chi F = i(S + FR_z),$$

$$\partial_\tau R = -\partial_\chi(E + 2aF),$$

$$\partial_\tau S = -\partial_\chi(F + 2aE),$$

$$\partial_\tau R_z = \frac{i}{2}(RF - SE), \tag{13}$$

which becomes equivalent to (7) by the reduction

$$F = E^*, \quad S = R^*, \quad R_z \text{ real}, \tag{14}$$

with the asterisk denoting complex conjugation. Note that the number a appearing in (13) and introduced in (8) measures anisotropy with $a \to 0$ in the isotropic limit.

III. THE LINEAR SYSTEM, SYMMETRIES AND CONSERVATION LAWS

The crucial first step to prove integrability in the sense of the inverse scattering (or spectral transform) method is to find some linear system

$$\partial_\tau \Phi = U\Phi, \quad \partial_\chi \Phi = V\Phi \tag{15}$$

such that the original equations of motion are the integrability conditions of this system. Such a system also provides the basis for establishing Darboux-Bäcklund transformations. Here we take U, V as 2×2 matrix functions defined as

$$U = \frac{1}{2}\begin{pmatrix} -i(\zeta^2 - a^2\zeta^{-2}) & E\zeta + aF/\zeta \\ F\zeta + aE/\zeta & i(\zeta^2 - a^2\zeta^{-2}) \end{pmatrix},$$

$$V = -\frac{1}{2m}\begin{pmatrix} i(\zeta^2 - a^2\zeta^{-2})R_z & R\zeta + aS/\zeta \\ S\zeta + aR/\zeta & -i(\zeta^2 - a^2\zeta^{-2})R_z \end{pmatrix}, \tag{16}$$

with $m = 1 + \zeta^2 + a^2\zeta^{-2}$ where ζ is an arbitrary complex number called the *spectral parameter*. $\Phi \equiv \Phi(\zeta, \tau, \chi)$ might be read as a two-component column vector or, as we prefer here, as a 2×2 matrix function. Then, indeed, the differential equations (13) are equivalent to the integrability condition $\partial_\chi U - \partial_\tau V + [U, V] = 0$ of the system (15). It should be noticed that in the isotropic limit $a \to 0$, the above matrices (16) coincide with the corresponding matrices in [6]. For our further procedure, however, it is of importance to assume $a > 0$.

Now we introduce $b = \sqrt{a}$ and take $\lambda = \zeta/b$ as the spectral parameter in order to get, instead of (16),

$$U = \frac{b}{2}\begin{pmatrix} -ib(\lambda^2-\lambda^{-2}) & E\lambda + F/\lambda \\ F\lambda + E/\lambda & ib(\lambda^2-\lambda^{-2}) \end{pmatrix},$$

$$V = \frac{b}{2w(\lambda)}Y,$$

$$Y = \begin{pmatrix} -ib(\lambda^2-\lambda^{-2})R_z & -R\lambda - S/\lambda \\ -S\lambda - R/\lambda & ib(\lambda^2-\lambda^{-2})R_z \end{pmatrix}, \tag{17}$$

$$w(\lambda) = 1 + b^2(\lambda^2+\lambda^{-2}). \tag{18}$$

From the linear system (15) and (17) one may easily derive a system of Riccati equations for the quotient $\alpha(\tau,\chi)=\Phi_{11}/\Phi_{21}$,

$$\partial_\tau\alpha = \frac{b}{2}[-2ib(\lambda^2-\lambda^{-2})\alpha + (E\lambda + F/\lambda) - (F\lambda + E/\lambda)\alpha^2],$$

$$\partial_\chi\alpha = \frac{b}{2w(\lambda)}[-2ib(\lambda^2-\lambda^{-2})R_z\alpha - (R\lambda + S/\lambda) + (S\lambda + R/\lambda)\alpha^2]. \tag{19}$$

We list three obvious symmetries of the matrix functions $U(\lambda), V(\lambda)$. In the subsequent relations the letter W represents either U or V. σ_1 and σ_3 are standard Pauli matrices:

$$\text{(i)}\quad W(1/\lambda) = \sigma_1 W(\lambda)\sigma_1, \tag{20}$$

$$\text{(ii)}\quad W(-\lambda) = \sigma_3 W(\lambda)\sigma_3. \tag{21}$$

Reduction. If (14) holds we get the additional symmetry $W^*(1/\lambda^*)=W(\lambda)$ or, when combined with (i),

$$\text{(iii)}\quad W^*(\lambda^*) = \sigma_1 W(\lambda)\sigma_1. \tag{22}$$

Correspondingly, with respect to the Riccati equations we get the following symmetries. (i) $\lambda\to 1/\lambda, \alpha\to 1/\alpha$. (ii) $\lambda\to-\lambda, \alpha\to-\alpha$. (iii) *Reduction*. $\lambda\to\lambda^*, \alpha\to 1/\alpha^*$.

The conservation law

$$\partial_\tau(R_x^2 + R_y^2 + R_z^2) = 0 \tag{23}$$

of (7) after the transformation (12) becomes

$$\partial_\tau\left[\frac{(R+S)^2}{4(1+2a)} - \frac{(R-S)^2}{4(1-2a)} + R_z^2\right] = 0. \tag{24}$$

The conservation law of energy is given by

$$\frac{1}{2}\partial_\chi(EF) + \partial_\tau R_z = 0. \tag{25}$$

We notice that the fourth and the fifth of the equations (13) are of the form of local conservation laws as well.

IV. DARBOUX TRANSFORMATION

Let us take only the U part of (15) with U given by the first equation of (17) and, preliminarily, ignore the dependence on χ. Thus we arrive at some type of a *scattering problem* with $E(\chi), F(\chi)$ being the *potentials*. Supposing some solution (Φ, U) is known, it is our goal to find a new solution $(\Phi^{[1]}, U^{[1]})$ with new potentials $E^{[1]}, F^{[1]}$. As an ansatz we take a gauge transformation,

$$\Phi^{[1]}(\lambda,\tau) = P(\lambda,\tau)\Phi(\lambda,\tau), \quad U^{[1]} = \partial_\tau P P^{-1} + PUP^{-1}, \tag{26}$$

with the matrix function P fulfilling

$$P(1/\lambda) = \sigma_1 P(\lambda)\sigma_1, \quad P(-\lambda) = \epsilon\sigma_3 P(\lambda)\sigma_3, \quad \epsilon = \pm 1. \tag{27}$$

Obviously, the symmetries (i) and (ii) are conserved under such a transformation.

If we take $P=P_{-1}/\lambda+P_0+P_1\lambda$ then from (27) we get $P_1\sigma_3=-\epsilon\sigma_3 P_1$. On the other hand, from the identity

$$U^{[1]}P - \partial_\tau P - PU = 0 \tag{28}$$

the highest order term $\propto\lambda^3$ gives $P_1\sigma_3=\sigma_3 P_1$. Thus we get $\epsilon=-1$. The simplest nontrivial matrix function fulfilling (27) with $\epsilon=-1$ is of the form

$$P(\lambda,\tau) = \begin{pmatrix} p_1(\tau)\lambda + p_{-1}(\tau)/\lambda & p_0(\tau) \\ p_0(\tau) & p_{-1}(\tau)\lambda + p_1(\tau)/\lambda \end{pmatrix}. \tag{29}$$

We require that $U^{[1]}$ should be of the form

$$U^{[1]}(\lambda,\tau) = \frac{1}{2}\begin{pmatrix} -i(\lambda^2-\lambda^{-2}) & E^{[1]}(\tau)\lambda + F^{[1]}(\tau)/\lambda \\ F^{[1]}(\tau)\lambda + E^{[1]}(\tau)/\lambda & i(\lambda^2-\lambda^{-2}) \end{pmatrix}. \tag{30}$$

Now we compare powers of λ in (28) where U from (17), $U^{[1]}$ from (30), and P from (29) have to be substituted. The coefficient of λ^{-3} vanishes. Comparison of the off-diagonal matrix elements of order λ^{-2} leads to

$$E^{[1]} = (2ibp_0 + p_1 E)/p_{-1} \quad\text{and}\quad F^{[1]} = (-2ibp_0 + p_{-1}F)/p_1. \tag{31}$$

Due to (27) and (29), the determinant $\det P=p_1 p_{-1}(\lambda^2+\lambda^{-2})+p_1^2+p_{-1}^2-p_0^2$ has four zeros λ_1, λ_1^{-1}, $-\lambda_1$, and $-\lambda_1^{-1}$. Consequently,

$$P(\lambda_1,\tau)\Phi(\lambda_1,\tau)q_1 = 0, \tag{32}$$

where q_1 is a nontrivial column vector. Both λ_1 and q_1 are independent of τ; see [12,13]. Then (32) may be rewritten

$$P(\lambda_1,\tau)\begin{pmatrix}\alpha_1(\tau)\\ 1\end{pmatrix} \equiv 0, \tag{33}$$

where α_1 may be characterized to be a solution to the Riccati equation (19) with $\lambda=\lambda_1$. We introduce the abbreviations

$$k_1 \equiv \alpha_1/\lambda_1 - \lambda_1/\alpha_1, \quad l_1 \equiv -\alpha_1\lambda_1 + 1/(\alpha_1\lambda_1). \tag{34}$$

Then, from (33), we get

$$\frac{p_1}{p_0} = \frac{k_1}{\lambda_1^2 - \lambda_1^{-2}}, \quad \frac{p_{-1}}{p_0} = \frac{l_1}{\lambda_1^2 - \lambda_1^{-2}}. \tag{35}$$

In order to complete the gauge transformation (26) a common τ-dependent factor of p_0, p_1, p_{-1} has to be determined. It does not influence the transformation of E,F; see (31). We may express p_1,p_{-1} by p_0 according to (35). Comparison of the coefficients of λ in the 12-element of (28) then determines the function $p_0(\tau)$ up to a constant multiplicative factor. With a suitable choice of this factor we get

$$p_0=(\lambda_1^2-\lambda_1^{-2})/\sqrt{k_1 l_1}, \quad p_1=1/p_{-1}=\sqrt{k_1/l_1}. \tag{36}$$

The determinant of P becomes independent of τ,

$$\det P(\lambda,\tau)=\lambda^2+\lambda^{-2}-\lambda_1^2-\lambda_1^{-2}. \tag{37}$$

Finally, one may confirm—most conveniently by using a PC—that the identity (28) is fulfilled.

Reduction. If for the seed solution $F=E^*$ holds and we take λ_1 of modulus 1 we may require that α_1 is real. Then from (31), (34), and (35) we see that $F^{[1]}=E^{[1]*}$ is fulfilled for the Darboux-transformed potential.

V. BÄCKLUND TRANSFORMATION

Now we switch on the χ dependence and wish to extend the Darboux transformation in order to find a transformation of the simultaneous system (15). We require $\alpha(\tau,\chi)$ to fulfill the simultaneous Riccati system (19), and we require that the matrix $V^{[1]}\equiv(\partial_\chi P+PV)P^{-1}$ take the same form as V. Then, with $V^{[1]}=(b/2w_1)Y^{[1]}, w_1=w(\lambda_1)$, we arrive at the identity

$$Y^{[1]}P-PY-(2w_1/b)\partial_\chi P=0. \tag{38}$$

Taking, at first, the 11 element and comparing powers of $\lambda^{-3},\lambda^{-1}$ and λ, respectively, we get

$$R_z^{[1]}=\left(1-\frac{2b^2}{w_1}p_0^2\right)R_z+\frac{ibp_0}{w_1}(p_{-1}S-p_1R),$$

$$R^{[1]}=\frac{1}{w_1}(p_1-2b^2p_{-1})(p_1R-2ibp_0R_z) + \left[1-\frac{1}{w_1}(1-2b^2p_{-1}^2)\right]S,$$

$$S^{[1]}=\frac{1}{w_1}(p_{-1}-2b^2p_1)(p_{-1}S+2ibp_0R_z) + \left[1-\frac{1}{w_1}(1-2b^2p_1^2)\right]R. \tag{39}$$

Finally it can be checked that with (39) the matrix identity (38) is fulfilled.

Reduction. If $F=E^*,S=R^*$ and R_z is real we may choose λ_1 of modulus 1 and take α_1 as real. Then we get $p_{-1}/p_0=-(p_1/p_0)^*$; $F^{[1]}=E^{[1]*},S^{[1]}=R^{[1]*}$ and real $R_z^{[1]}$.

VI. THE *N*-FOLD BÄCKLUND TRANSFORMATION

On the base of Secs. IV and V we establish compact algebraic formulas for an N-fold Bäcklund transformation and specify to the simplest examples.

A. Transformation of the field

For the N-fold iterated Bäcklund transformation we use the ansatz

$$\Phi^{[N]}=P^{[N]}\Phi, \quad P^{[N]}=\sum_{r=-N}^{N}P_r\lambda^r. \tag{40}$$

The identity (28) now becomes

$$U^{[N]}P^{[N]}-P_\chi^{[N]}-P^{[N]}U=0. \tag{41}$$

The terms $\propto\lambda^N$ in (41) then give $\sigma_3P_N-P_N\sigma_3=0$. Therefore the symmetry relations (27) now read

$$P^{[N]}(1/\lambda)=\sigma_1P^{[N]}(\lambda)\sigma_1, \quad P^{[N]}(-\lambda)=(-1)^N\sigma_3P^{[N]}(\lambda)\sigma_3. \tag{42}$$

To save space we will treat the case of even $N,N=2n$, only. The case $N=2n-1$ can be treated analogously. Then we get the structure

$$P_{2k}=\begin{pmatrix} p_{2k} & 0 \\ 0 & p_{-2k}\end{pmatrix}, \quad k=-n,\ldots,n, \tag{43}$$

$$P_{2l+1}=\begin{pmatrix} 0 & p_{2l+1} \\ p_{-2l-1} & \end{pmatrix}, \quad l=-n,\ldots,n-1. \tag{44}$$

We assume that N solutions $\alpha_j(\chi,\tau),\lambda_j$ of the Riccati equations (19) are known and require

$$P^{[N]}(\lambda_j,\chi,\tau)\begin{pmatrix}\alpha_j\\ 1\end{pmatrix}\equiv 0, \quad j=1,\ldots,N. \tag{45}$$

These are $2N$ equations which may be combined by use of the notation $\lambda_{N+j}=\lambda_j^{-1},\alpha_{N+j}=\alpha_j^{-1},\beta_a=\alpha_a^{-1},j=1,2\ldots,N,a=1,2\ldots,2N$. Then we arrive at the linear algebraic system of $2N$ equations for $2N+1$ unknowns $p_{-N},\ldots,p_N$,

$$\sum_{k=-n}^{n}p_{2k}\lambda_a^{2k}+\beta_a\sum_{l=-n}^{n-1}p_{2l+1}\lambda_a^{2l+1}=0, \quad a=1,2\ldots,2N. \tag{46}$$

The solution may be given in terms of Vandermonde-like determinants; see the Appendix,

$$\frac{p_{2n}}{p_0}=(-1)^n\frac{\mathcal{V}_{2n,2n}(\lambda_a^{-2(n-1)};\beta_a\lambda_a^{-2n+1}|\lambda_a^2)}{\mathcal{V}_{n,n,2n}(\lambda_a^{-2n};\lambda_a^2;\beta_a\lambda_a^{-2n+1}|\lambda_a^2)}; \tag{47}$$

$$\frac{p_{2n-1}}{p_0}=(-1)^n\frac{\mathcal{V}_{2n+1,2n-1}(\lambda_a^{-2(n-1)};\beta_a\lambda_a^{-2n+1}|\lambda_a^2)}{\mathcal{V}_{n,n,2n}(\lambda_a^{-2n};\lambda_a^2;\beta_a\lambda_a^{-2n+1}|\lambda_a^2)}, \tag{48}$$

$$p_{-r}=p_r(\lambda_a\to\lambda_a^{-1},\beta_a\to\alpha_a). \tag{49}$$

To find the potential we take the terms $\propto\lambda^N$ in the off-diagonal part of (41) and get

$$E^{[N]}=\frac{2ibp_{N-1}+p_NE}{p_{-N}}, \tag{50}$$

$$F^{[N]}=\frac{-2ibp_{1-N}+p_{-N}F}{p_N}. \tag{51}$$

B. Transformation of the atomic variables

For $N=2n$ let us write

$$P^{[N]} = \sum_{j=-N}^{N} P_j \lambda^j = \begin{pmatrix} A(\lambda) & B(\lambda) \\ B(1/\lambda) & A(1/\lambda) \end{pmatrix}, \tag{52}$$

$$A(\lambda) = \sum_{k=-n}^{n} p_{2k}\lambda^{2k}, \quad B(\lambda) = \sum_{l=-n}^{n-1} p_{2l+1}\lambda^{2l+1}. \tag{53}$$

In order to determine $A(\lambda)$ we may read the first of equations (53) together with (46) as a homogeneous linear algebraic system for $A(\lambda)$ and p_a and find

$$A(\lambda) = (-1)^n \frac{\mathcal{V}_{2n+1,2n}(\lambda_a^{-2n},\lambda^{-2n};\beta_a\lambda_a^{-2n+1},0|\lambda_a^2,\lambda^2)}{\mathcal{V}_{n,n,2n}(\lambda_a^{-2n};\lambda_a^2;\beta_a\lambda_a^{-2n+1}|\lambda_a^2)} p_0. \tag{54}$$

Analogously we get

$$B(\lambda) = (-1)^n \frac{\mathcal{V}_{2n+1,2n}(\lambda_a^{-2n},0;\beta_a\lambda_a^{-2n+1},\lambda^{-2n+1}|\lambda_a^2,\lambda^2)}{\mathcal{V}_{n,n,2n}(\lambda_a^{-2n};\lambda_a^2;\beta_a\lambda_a^{-2n+1}|\lambda_a^2)} p_0. \tag{55}$$

The matrix V is transformed according to $V^{[N]} = P^{[N]}V(P^{[N]})^{-1} + (2w/b)P_\chi^{[N]}(P^{[N]})^{-1}$. Then for the matrix

$$Z \equiv \frac{2w}{b}V = \begin{pmatrix} -ib(\lambda^2-\lambda^{-2})R_z & -R\lambda - S/\lambda \\ -S\lambda - R/\lambda & ib(\lambda^2-\lambda^{-2})R_z \end{pmatrix} \tag{56}$$

together with the specification that λ is taken at a zero λ_0 of $w(\lambda) \equiv 1+b^2(\lambda^2+\lambda^{-2})$ we get the transformation formula

$$Z^{[N]}(\lambda_0) = P^{[N]}(\lambda_0)Z(\lambda_0)[P^{[N]}(\lambda_0)]^{-1}. \tag{57}$$

With the further specification that we start from vacuum $R=S=0, R_z=-1$ we arrive at

$$R_z^{[N]} = -[A(\lambda_0)A(\lambda_0^{-1}) + B(\lambda_0)B(\lambda_0^{-1})]/D,$$

$$R^{[N]} = 2ib[\lambda_0 A(\lambda_0)B(\lambda_0) + \lambda_0^{-1}A(\lambda_0^{-1})B(\lambda_0^{-1})]/D, \tag{58}$$

$$S^{[N]} = -2ib[\lambda_0^{-1}A(\lambda_0)B(\lambda_0) + \lambda_0 A(\lambda_0^{-1})B(\lambda_0^{-1})]/D,$$

$$D = A(\lambda_0)A(\lambda_0^{-1}) - B(\lambda_0)B(\lambda_0^{-1}). \tag{59}$$

Reduction. Under the assumption that the symmetry (14) is fulfilled for the seed solution we assume that all $\lambda_j, j=1,\dots,N$ either are of modulus 1 or appear as conjugate complex pairs, $\lambda_l=\lambda_k^*$. We take either α_j as real or pairwise $\alpha_l=1/\alpha_k^*$ respectively. Then it holds that $A(1/\lambda_0)=A^*(\lambda_0)$ and $B(1/\lambda_0)=-B^*(\lambda_0)$, and the symmetry (14) is conserved under the N-fold Bäcklund transformation.

VII. EXAMPLES

Here we will specify the simplest soliton solutions only. Principally, it has been demonstrated [14,15] that formulas written in terms of Vandermonde-like determinants such as those given in Sec. VI are well suited for numerical evaluation of higher-order solutions.

A. The soliton

Now we start from vacuum $E=F=R=S=0, R_z=-1$ and take $\lambda_1=i\exp(i\varphi)$. The simultaneous Riccati equations (19) admit the real solution

$$\alpha_1 = \exp[\mu_1(-\tau+\chi/w_1)] \tag{60}$$

with

$$\mu_1 = 2b^2\sin 2\varphi, \quad w_1 = 1-2b^2\cos 2\varphi. \tag{61}$$

Then, according to (31) and (35), we get the soliton solution

$$E^{[1]} = \frac{i\mu_1}{b}\operatorname{sech}[\mu_1(\tau-\chi/w_1) - i\varphi], \quad F^{[1]} = E^{[1]*},$$

$$R^{[1]} = (E^{[1]} + b^2E^{[1]*})/w_1, \quad S^{[1]} = R^{[1]*},$$

$$R_z^{[1]} = -1 + |E^{[1]}|^2/2w_1. \tag{62}$$

The pulse shape may be written as

$$|E^{[1]}|^2 = \frac{(\mu_1/b)^2}{\sinh^2[\mu_1(\tau-\chi/w_1)] + \cos^2\varphi}. \tag{63}$$

B. The breather

Again we start from the vacuum as the seed solution, but now we choose $N=2$ and $\lambda_2=\lambda_1^*$. According to [6] solutions of particular interest are expected for $\zeta_1 \equiv b\lambda_1$ near to the imaginary unit. Therefore we write

$$\lambda_1 = i\frac{\rho}{b}e^{i\varphi}, \quad \lambda_2=\lambda_1^*, \quad \lambda_3=\lambda_1^{-1}, \quad \lambda_4=\lambda_2^{-1}. \tag{64}$$

The respective functions $\beta(\chi,\tau)$ may be written

$$\beta_1 = e^{\beta_r+i\theta_i}, \quad \beta_2 = e^{-\theta_r+i\theta_i}, \quad \beta_3=\beta_1^{-1}=\alpha_1, \quad \beta_4=\beta_2^{-1}=\alpha_2, \tag{65}$$

where θ_r and θ_i denote real-valued linear functions of χ and τ which are determined by

$$\theta_r + i\theta_i = \mu_1(\tau-\chi/w_1) \tag{66}$$

with the complex numbers

$$\mu_1 = i[-\rho^2e^{2i\varphi} + (b^4/\rho^2)e^{-2i\varphi}], \tag{67}$$

$$w_1 = 1 - \left[\rho^2e^{2i\varphi} + (b^4/\rho^2)e^{-2i\varphi}\right]. \tag{68}$$

Then, according to (47)–(50), we find the solution

$$E^{[2]} = -\frac{ip_1}{p_{-2}} = -\frac{iD_1}{D_{-2}} \tag{69}$$

with the 4×4 determinants D_1, D_{-2} being defined as follows:

$$D_1 = \begin{vmatrix} \lambda_1^{-2} & 1 & \lambda_1^2 & \beta_1\lambda_1^{-1} \\ \lambda_2^{-2} & 1 & \lambda_2^2 & \beta_2\lambda_2^{-1} \\ \lambda_3^{-2} & 1 & \lambda_3^2 & \beta_3\lambda_3^{-1} \\ \lambda_4^{-2} & 1 & \lambda_4^2 & \beta_4\lambda_4^{-1} \end{vmatrix}, \tag{70}$$

$$D_{-2} = \begin{vmatrix} 1 & \lambda_1^2 & \beta_1\lambda_1^{-1} & \beta_1\lambda_1 \\ 1 & \lambda_2^2 & \beta_2\lambda_2^{-1} & \beta_2\lambda_2 \\ 1 & \lambda_3^2 & \beta_3\lambda_3^{-1} & \beta_3\lambda_3 \\ 1 & \lambda_4^2 & \beta_4\lambda_4^{-1} & \beta_4\lambda_4 \end{vmatrix}. \tag{71}$$

We write down explicitly the electric field of the general breather solution, and we give its approximation under the assumption $a < \varphi \ll 1$,

$$\begin{aligned} E^{[2]} &= -2i\rho \sin(2\varphi) \\ &\quad \times \frac{e^{-i\theta_i}\cosh(\theta_r + i\varphi) + (a/\rho^2)e^{i\theta_i}\cosh(\theta_r - 3i\varphi)}{\cosh^2(\theta_r - i\varphi) - (a/\rho^2)e^{-2i\theta_i}\sin^2(2\varphi)} \\ &= -2i\rho \sin(2\varphi)e^{-i\theta_i}\frac{\cosh(\theta_r + i\varphi)}{\cosh^2(\theta_r - i\varphi)} \\ &\quad + e^{i\theta_i}\frac{4ia\varphi}{\rho \cosh \theta_r} + O(a\varphi^2). \end{aligned} \tag{72}$$

The formulas describing the atomic state of the general breather solution would become rather lengthy. Therefore, here we write down only the approximate solution for $a < \varphi \ll 1$ and for exact resonance, i.e., $\rho = 1$,

$$R_z^{[2]} = \frac{\cos^2\varphi - \sinh^2\theta_r}{\cos^2\varphi + \sinh^2\theta_r} + 8a\varphi\frac{\sin(2\theta_i)\sinh \theta_r}{\cosh^3\theta_r} + O(a\varphi^2), \tag{73}$$

$$\begin{aligned} R^{[2]} &= -2\mathrm{e}^{-i\theta_i}\frac{\cos \varphi \sinh \theta_r}{\cosh^2(\theta_r - i\varphi)} \\ &\quad - 2ae^{i\theta_i}\left[\frac{\sinh \theta_r}{\cosh^2\theta_r} + i\varphi\frac{3\cosh \theta_r - \cosh 3\theta_r}{\cosh^4\theta_r}\right] \\ &\quad + O(a\varphi^2). \end{aligned} \tag{74}$$

In the isotropic limit $a \to 0$ we recover the corresponding solution given in [6]. In this limit and with ρ arbitrary we find the intensity

$$|E^{[2]}|^2 = \frac{4\rho^2\sin^2(2\varphi)}{\sinh^2[\theta_r] + \cos^2\varphi} \tag{75}$$

which is of the same shape as the single soliton (63). For $\varphi \ll 1$, i.e., λ_1 near to the imaginary axis, the intensity is approximately of sech^2 shape while for φ near to $\pi/2$, i.e., λ_1 near to the real axis, it is approximately of Lorentzian shape.

We state that the breather solution for $a \ll 1, \varphi \ll 1$ approximates the one-soliton solution of the SIT equations in agreement with [6] and analogous to the plane wave RMB equations [1,2].

VIII. SUMMARY AND CONCLUSIONS

We have found a new integrable system and established Bäcklund transformations and N-soliton formulas for this system. This is another example that N-soliton formulas can be written in a rather compact and efficient way by use of Vandermonde-type determinants. The one-soliton formula contains a free parameter which interpolates between sech^2 and Lorentzian shapes, [see (63)], and the same holds for the breather shape in the isotropic limit; see (75).

It might be of interest to point out structural differences between the soliton manifolds of (i) the SIT equations [16], (ii) the plane wave RMB equations [1,2], (iii) the rotating RMB equations [6], and (iv) the rotating RMB equations with anisotropy as presented here.

The application of a one-step Bäcklund transformation to vacuum generates a soliton in the usual sense for (i), while for equations (ii) and (iv) half-cycle pulses are generated. For our equations (iv), see the first of Eqs. (62), the amplitude of this pulse is going up and down while its phase moves from $\phi_0 = 2a\varphi \sin(2\varphi)$ to $-\phi_0$. In the isotropic limit $a \to 0$ the maximum amplitude is going down while the pulse width becomes infinite. The physical relevance of pulses with less than one cycle is supported by some literature; see [17] and articles quoted therein. In case (iii) a Bäcklund transformation applied to vacuum generates a harmonic wave.

It is a common feature of equations (ii), (iii), and (iv) that some limits of the breather solutions—generated by two-step Bäcklund transformations—correspond to the one-soliton solution of the SIT equations (i).

In equations (i) and (ii) inhomogeneous broadening can be included without destroying integrability. This, probably, cannot be done for (iii) and (iv).

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APPENDIX: VANDERMONDE-LIKE DETERMINANTS

The notion of Vandermonde-like determinants $\mathcal{V}_{n_1 n_2}$ with two subscripts as it has been introduced [18] and applied [6,14,15,20,21] is generalized in a straightforward way. In the present paper such determinants with two and with three subscripts are used. We also generalize the so-called reduction formula expressing any Vandermonde-like determinant as a sum over products of genuine Vandermonde determinants.

Definition. A Vandermonde-like determinant of type $(n_1, n_2, \ldots, n_m), \Sigma n_j = n$ is the determinant of an $n \times n$ matrix M consisting of m rectangular submatrices where the submatrix of number j has the dimension $n\times n_j$, and there is an n-tupel of numbers $x_r, r=1,\ldots,n$ such that within each of these submatrices

$$M_{r,s+1}=x_r M_{rs} \tag{A1}$$

holds. The matrix M and, subsequently, its determinant are characterized by the first columns of the submatrices, i.e., by some $n\times m$ matrix $a_{rj}, r=1,\ldots,n, j=1,\ldots,m$ together with the x_r.

Definition [19,22]. A permutation p of the natural numbers $1,2,\ldots,n$ is called a shuffle of type $(n_1,n_2,\ldots,n_m), \Sigma n_j=n$ or an element of $Sh(n_1,n_2,\ldots,n_m)$ iff it is increasing on each of the m subsets $\mathcal{S}_j=\{\Sigma_1^{j-1}n_k+1,\ldots,\Sigma_1^j n_k\}, j=1,\ldots,m$, i.e., $p(r)<p(s)$ for $r<s$ and $r,s\in\mathcal{S}_j$.

Reduction formula:

$$\begin{aligned}\mathcal{V}_{n_1 n_2\ldots n_m}&(a_{1r};a_{2r};\cdots a_{mr}|x_r)\\ &=\sum_{p\in Sh(n_1,\ldots,n_m)}\mathrm{index}(p)\left(\prod_{j=1}^{n_1}a_{1p(j)}\right)\\ &\quad\times\left(\prod_{k=n_1+1}^{n_1+n_2}a_{2p(k)}\right)\cdots\left(\prod_{l=n-n_m}^{n}a_{mp(l)}\right)\\ &\quad\times\mathcal{V}_{n_1}(x_{p(1)},\ldots,x_{p(n_1)})\mathcal{V}_{n_2}(x_{p(n_1+1)},\ldots,x_{p(n_1+n_2)})\cdots\\ &\quad\times\mathcal{V}_{n_m}(x_{p(n-n_m+1)},\ldots,x_{p(n)}).\end{aligned} \tag{A2}$$

Here $\mathcal{V}_s(x_1,\ldots,x_s)$ denotes a usual Vandermonde determinant, i.e., the determinant of an $s\times s$ matrix whose kth row is $(x_k,x_k^2,\ldots,x_k^{s-1})$ and which is equal to the product of ordered differences.

The proof of the reduction formula is almost obvious: One only has to develop the nth order determinant as a sum over m-fold products of determinants of orders $n_1,n_2,\ldots,n_m$.

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